1.a) $z = x^2 + y^2 + xy$, $x = \text{sen}\,t$, $y = e^t$

Recordamos la regla de la cadena.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Reemplazamos directamente

$$\frac{\partial z}{\partial t} = (2x + y)\cos t + (2y + x)\,e^t$$

$$= (2\,\text{sen}\,t + e^t)\cos t + (2e^t + \text{sen}\,t)\,e^t$$

1.b) $z = e^{y/x}$, $x = t^2$, $y = 1 + 2t$. Calculamos directamente

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \left(\frac{-y}{x^2}e^{y/x}\right)\cdot 2t + \left(\frac{1}{x}e^{y/x}\right)\cdot 2$$

$$= \left(\frac{-(1+2t)}{t^4}e^{\frac{1+2t}{t^2}}\right)2t + \left(\frac{1}{t^2}e^{\frac{1+2t}{t^2}}\right)\cdot 2$$

$$= -2\,\frac{1+2t}{t^3}e^{\frac{1+2t}{t^2}} + \frac{2}{t^2}e^{\frac{1+2t}{t^2}}$$

2.a) $z = \text{arcsen}(x - y)$, $x = s^2 + t^2$, $y = 1 - 2st$

Recordando que

$$\frac{d}{dx}\text{arcsen}(x) = \frac{1}{\sqrt{1-x^2}}$$

Tenemos que

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left(\frac{1}{\sqrt{1-(x-y)^2}}\right)(2t) + \left(\frac{-1}{\sqrt{1-(x-y)^2}}\right)(-2s)$$

$$= \frac{2(t+s)}{\sqrt{1-(x-y)^2}}$$

Veamos que ocurre con el denominador

$$1-(x-y)^2 = 1-(s^2+t^2-1+2st)^2$$
$$= 1-((s+t)^2-1)^2$$
$$= (1+((s+t)^2-1))(1-((s+t)^2-1))$$
$$= (s+t)^2(2-(s+t)^2)$$

Luego

$$\frac{\partial z}{\partial t} = \frac{2(s+t)}{\sqrt{(s+t)^2(2-(s+t)^2)}}$$

$$= \frac{2(s+t)}{(s+t)\sqrt{2-(s+t)^2}} = \frac{2}{\sqrt{2-(s+t)^2}}$$

Para s, en los primeros pasos podemos darnos cuenta que

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial s}$$

2. b) $z = e^r \cos\theta$, $r = st$, $\theta = \sqrt{s^2+t^2}$.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

$$= (e^r \cos\theta)s + (-e^r \operatorname{sen}\theta)\frac{s}{\sqrt{s^2+t^2}}$$

$$= e^{st}s\left(\cos\left(\sqrt{s^2+t^2}\right) - \frac{\operatorname{sen}\left(\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\right)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$= (e^r \cos\theta)t + (-e^r \operatorname{sen}\theta)\frac{t}{\sqrt{s^2+t^2}}$$

$$= e^{st}t\left(\cos\left(\sqrt{s^2+t^2}\right) - \frac{\operatorname{sen}\left(\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\right) //$$

3) $g(u,v) = f(x,y)$, $x = e^u + \operatorname{sen} v$, $y = e^u + \cos v$.

Calculamos directamente las primeras derivadas

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$= f_x \cdot e^u + f_y e^u = e^u(f_x(x,y))$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$= f_x \cos v - f_y \operatorname{sen} v$$

Luego, notamos que cuando $u = v = 0 \Rightarrow x = 1, y = 2$

$$g_u(0,0) = f_u(1,2) = e^0(f_x(1,2) + f_y(1,2))$$

$$= 2 + 5 = 7 //$$

$$g_v(0,0) = f_v(1,2) = f_x(1,2)\cos(0) - f_y(1,2)\operatorname{sen}(0)$$

$$= 2 //$$

4) Para esto utilizaremos las siguientes variables.

$$u = x + at \quad , \quad v = x - at.$$

$$\Rightarrow z = f(u) + g(v)$$

Notamos que

$$\frac{\partial u}{\partial x} = 1 \quad , \quad \frac{\partial v}{\partial x} = 1 \quad ,$$

$$\frac{\partial u}{\partial t} = a \quad , \quad \frac{\partial v}{\partial t} = -a$$

Luego utilizamos regla de la cadena para calcular $\frac{\partial^2 z}{\partial t^2}$, vamos por paso

- $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t}$$

$$= \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial t}$$

$$= \frac{\partial f}{\partial u}\, a + \frac{\partial g}{\partial v}(-a) = a(f_u - g_v)$$

- $$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}(a(f_u - g_v))$$

$$= a\left(\frac{\partial}{\partial t} f_u - \frac{\partial}{\partial t} g_v\right)$$

$$= a\left(\frac{\partial f_u}{\partial u}\cdot\frac{\partial u}{\partial t} - \frac{\partial g_v}{\partial v}\cdot\frac{\partial v}{\partial t}\right)$$

regla de la cadena para f_u y g_v

$$= a\left(\frac{\partial f_u}{\partial u}(a) - \frac{\partial g_v}{\partial v}\cdot(-a)\right) = a^2(f_{uu} + g_{vv})$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2} = a^2(f_{uu} + g_{vv})$$

Ahora comprobamos

- $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$

$= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$

$= f_u + g_v$

- $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

$= \frac{\partial}{\partial x}(f_u + g_v)$

$= \frac{\partial}{\partial x} f_u + \frac{\partial}{\partial x} g_v$

$= \frac{\partial f_u}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g_v}{\partial v}\frac{\partial v}{\partial x}$

$= f_{uu} + g_{vv}$

$\Rightarrow \frac{\partial^2 z}{\partial x^2} = f_{uu} + g_{vv}$ //

comprobamos que

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$ //