PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
FACULTAD DE MATEMATICAS
DEPARTAMENTO DE MATEMATICA
Segundo semestre 2020

Ayudantía 15 - MAT1610

1. Determine $\underbrace{\int \cos^2(8\pi x)\text{sen}^2(5x)dx}_{I}$.

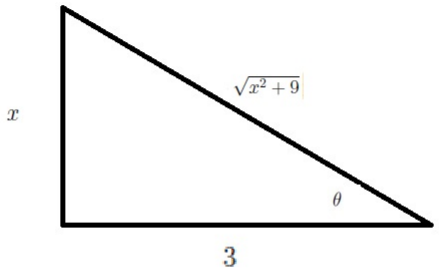
Solución:
Reescribir usando primero la identidad $\cos(\alpha)\text{sen}(\beta) = \frac{\text{sen}(\alpha+\beta)-\text{sen}(\alpha-\beta)}{2}$ y después que $\text{sen}^2(\theta) = \frac{1-\cos(2\theta)}{2}$ y $\text{sen}(\alpha)\text{sen}(\beta) = \frac{cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$. Entonces,

$$
\begin{aligned}
I &= \int \left(\cos(8\pi x)\text{sen}(5x)\right)^2 dx \\
&= \int \left(\frac{\text{sen}((8\pi+5)x) - \text{sen}((8\pi-5)x)}{2}\right)^2 dx \\
&= \frac{1}{4}\int \left[\text{sen}^2((8\pi+5)x) - 2\text{sen}((8\pi+5)x)\text{sen}((8\pi-5)x) + \text{sen}^2((8\pi-5)x)\right] dx \\
&= \frac{1}{4}\int \text{sen}^2((8\pi+5)x)dx - \frac{1}{2}\int \text{sen}((8\pi+5)x)\text{sen}((8\pi-5)x)dx \\
&+ \frac{1}{4}\int \text{sen}^2((8\pi-5)x)dx \\
&= \frac{1}{8}\int \left(1-\cos(2(8\pi+5)x)\right) dx - \frac{1}{4}\int \left(\cos(10x) - \cos(16\pi x)\right) dx \\
&+ \frac{1}{8}\int \left(1-\cos(2(8\pi-5)x)\right) dx \\
&= \frac{1}{8}\left(x - \frac{\text{sen}(2(8\pi+5)x)}{2(8\pi+5)}\right) - \frac{1}{4}\left(\frac{\text{sen}(10x)}{10} - \frac{\text{sen}(16\pi x)}{16\pi}\right) \\
&+ \frac{1}{8}\left(x - \frac{\text{sen}(2(8\pi-5)x)}{2(8\pi-5)}\right) + C
\end{aligned}
$$

2. Determine $\underbrace{\int \frac{x^2}{\sqrt{x^2+9}}dx}_{I}$

Solución:

Considerar la sustitución $x = 3\tan(\theta)$ o $\frac{x}{3} = \tan(\theta)$ con $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ entonces $dx = 3\sec^2(\theta)d\theta$ y $\sqrt{x^2+9} = \sqrt{9\tan^2(\theta)+9} = \sqrt{9(\tan^2(\theta)+1)} = 3|\sec(\theta)| = 3\sec(\theta)$
$\sec(\theta) > 0$ porque $\cos(\theta) > 0$ para $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.



$$\begin{aligned} I = \int \frac{x^2}{\sqrt{x^2+9}}dx &= \int \frac{9\tan^2(\theta)3\sec^2(\theta)d\theta}{\sqrt{9(\tan^2(\theta)+1)}} \\ &= 9\underbrace{\int \tan^2(\theta)\sec(\theta)d\theta}_{I_1} \end{aligned}$$

Una forma para determinar I_1 es aplicar integración por partes

$$\begin{aligned} I_1 &= \int \tan^2(\theta)\sec(\theta)d\theta \\ &= \int \underbrace{\tan(\theta)}_{u}\underbrace{\tan(\theta)\sec(\theta)d\theta}_{dv} \\ &= \tan(\theta)\sec(\theta) - \int \sec(\theta)\sec^2(\theta)d\theta \\ &= \tan(\theta)\sec(\theta) - \int \sec(\theta)\left(\tan^2(\theta)+1\right)d\theta \\ &= \tan(\theta)\sec(\theta) - \underbrace{\int \sec(\theta)\tan^2(\theta)d\theta}_{I_1} - \int \sec(\theta)d\theta \\ &= \tan(\theta)\sec(\theta) - I_1 - \ln\left(|\sec(\theta)+\tan(\theta)|\right) \end{aligned}$$

Por lo tanto,

$$2I_1 = \tan(\theta)\sec(\theta) - \ln\left(|\sec(\theta) + \tan(\theta)|\right)$$

es decir,

$$I_1 = \frac{\tan(\theta)\sec(\theta) - \ln\left(|\sec(\theta) + \tan(\theta)|\right)}{2} + K$$

Así,

$$\begin{aligned} I &= 9I_1 \\ &= \frac{9}{2}\left(\tan(\theta)\sec(\theta) - \ln\left(|\sec(\theta) + \tan(\theta)|\right)\right) + C \\ &= \frac{9}{2}\left(\frac{x}{3}\frac{\sqrt{x^2+9}}{3} - \ln\left(\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right|\right)\right) + C \\ &= \frac{x\sqrt{x^2+9}}{2} - \frac{9}{2}\ln\left(\left|\frac{x+\sqrt{x^2+9}}{3}\right|\right) + C \end{aligned}$$

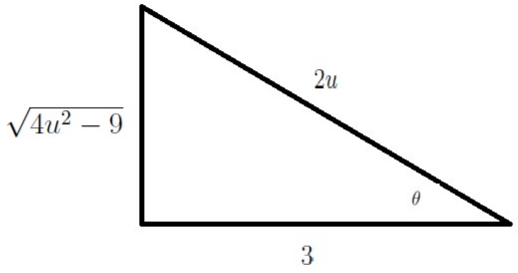
3. Determine $\underbrace{\int \frac{1}{\sqrt{(4x^2-24x+27)^3}}dx}_{I}$.

Solución:

Note que, $\int \frac{1}{\sqrt{(4x^2-24x+27)^3}}dx = \int \frac{1}{\sqrt{(4(x-3)^2-9)^3}}dx$. Entonces, consideranto la sustitución $u = x-3$ se tiene que $du = dx$ y

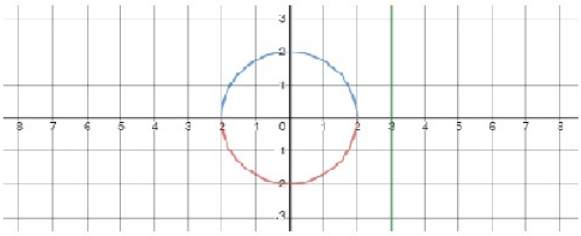
$$\int \frac{1}{\sqrt{(4x^2-24x+27)^3}}dx = \int \frac{1}{\sqrt{(4u^2-9)^3}}du$$

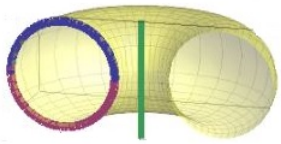
Ahora, haciendo la sustitución $\frac{2}{3}u = \sec(\theta)$, $0 \leq \theta < \frac{\pi}{2}$ o $\pi \leq \theta < \frac{3\pi}{2}$, entonces
$du = \frac{3}{2}\sec(\theta)\tan(\theta)d\theta$



$$
\begin{aligned}
I &= \int \frac{1}{\sqrt{(4u^2-9)^3}}du \\
&= \int \frac{1}{\sqrt{9^3\left(\left(\frac{2}{3}u\right)^2-1\right)^3}}du \\
&= \frac{1}{27}\int \frac{1}{\sqrt{(\sec^2(\theta)-1)^3}}\frac{3}{2}\sec(\theta)\tan(\theta)d\theta \\
&= \frac{1}{18}\int \frac{\sec(\theta)\tan(\theta)d\theta}{\tan^3(\theta)} \\
&= \frac{1}{18}\int \frac{\sec(\theta)d\theta}{\tan^2(\theta)} \\
&= \frac{1}{18}\int \frac{\cos(\theta)d\theta}{\operatorname{sen}^2(\theta)} \\
&= -\frac{1}{18}\frac{1}{\operatorname{sen}(\theta)}+C \\
&= -\frac{1}{18}\frac{2u}{\sqrt{4u^2-9}}+C \\
&= -\frac{1}{18}\frac{2(x-3)}{\sqrt{4(x-3)^2-9}}+C \\
&= -\frac{1}{9}\frac{(x-3)}{\sqrt{4(x-3)^2-9}}+C
\end{aligned}
$$

4. Hallar el volumen del toro generado en la rotación del círculo $x^2+y^2=4$ alrededor de la recta $x=3$.
Solución:
Idea gráfica





$$\begin{aligned}
V = &= \int_{-2}^{2} 2\pi(3-x)\left(\sqrt{4-x^2}-\left(-\sqrt{4-x^2}\right)\right)dx \\
&= 4\pi\int_{-2}^{2}(3-x)\sqrt{4-x^2}dx \\
&= 4\pi\left(3\underbrace{\int_{-2}^{2}\sqrt{4-x^2}dx}_{I_1}-\underbrace{\int_{-2}^{2}x\sqrt{4-x^2}dx}_{I_2}\right) \\
&= 4\pi\left(3I_1-I_2\right) \\
&= 12\pi I_1
\end{aligned}$$

Note que el integrando de la integral I_2 es impar y el intervalo de integración es simétrico respecto al origen entonces la integral vale cero (Para calcular I_2 considere la sustitución $u=4-x^2$, entonces $du=-2xdx$ o $-\frac{du}{2}=xdx$ y si $x=-2$ entonces $u=0$ y si $x=2$ entonces $u=0$).
Para calcular I_1 considere la sustitución $x=2\text{sen}(\theta)$, con $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$ entonces,
$dx=2\cos(\theta)d\theta$, $\sqrt{4-x^2}=\sqrt{4-4\text{sen}^2(\theta)}=2|\cos(\theta)|=2cos(\theta)$, $\theta=\text{arcsen}\left(\frac{x}{2}\right)$ y si $x=0$ entonces $\theta=\text{arcsen}\left(\frac{0}{2}\right)=0$ y si $x=2$ entonces $u=\text{arcsen}\left(\frac{2}{2}\right)=\text{arcsen}\,(1)=\frac{\pi}{2}$.

Por lo tanto,

$$\begin{aligned}
I_1 &= \int_{-2}^{2} \sqrt{4-x^2}dx \\
&= 2\int_{0}^{2} \sqrt{4-x^2}dx \\
&= 2\int_{0}^{\frac{\pi}{2}} 2cos(\theta)2\cos(\theta)d\theta \\
&= 8\int_{0}^{\frac{\pi}{2}} \cos^2(\theta)d\theta \\
&= 8\int_{0}^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}d\theta \\
&= 4\int_{0}^{\frac{\pi}{2}} (1+\cos(2\theta))d\theta \\
&= 4\left(\theta + \frac{\text{sen}(2\theta)}{2}\right)\Big|_0^{\frac{\pi}{2}} \\
&= 4\theta + 2\text{sen}(2\theta)|_0^{\frac{\pi}{2}} \\
&= 2\pi
\end{aligned}$$

Así $V = 12\pi \cdot 2\pi = 24\pi^2$ unidades de volumen.

5. Considere la sustitución $u = \sqrt{\frac{1-x}{1+x}}$ en

$$\underbrace{\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}dx}_{I}$$

y resuelva.

Solución:

Sea $u = \sqrt{\frac{1-x}{1+x}}$ entonces,

$du = \frac{1}{2\sqrt{\frac{1-x}{1+x}}}\left(\frac{1-x}{1+x}\right)'dx = \frac{1}{2\sqrt{\frac{1-x}{1+x}}}\left(-1+\frac{2}{1+x}\right)'dx = \frac{1}{\sqrt{\frac{1-x}{1+x}}}\left(-\frac{1}{(1+x)^2}\right)dx$

Note que $u = \sqrt{\frac{1-x}{1+x}} = \sqrt{-1+\frac{2}{1+x}}$, entonces $u^2 = -1+\frac{2}{1+x}$, $\frac{u^2+1}{2} = \frac{1}{1+x}$ ó $\frac{(u^2+1)^2}{4} = \frac{1}{(1+x)^2}$, $x = \frac{2}{u^2+1} - 1 = \frac{1-u^2}{u^2+1}$ y $x^2 = \frac{(1-u^2)^2}{(u^2+1)^2}$. Por lo tanto,

$du = -\frac{1}{u}\frac{(u^2+1)^2}{4}dx$ o $dx = -\frac{4u}{(u^2+1)^2}du$ y

$$\begin{aligned}
I = &= -\int u\frac{(u^2+1)^2}{(1-u^2)^2}\frac{4u}{(u^2+1)^2}du \\
&= -4\int \frac{u^2}{(1-u^2)^2}du \\
&= -4\int \frac{u^2}{(1-u)^2(1+u)^2}du \\
&= -4\left[\int \frac{A}{1-u}du + + \int \frac{B}{(1-u)^2}du + \int \frac{C}{1+u}du + \int \frac{D}{(1+u)^2}du\right] \\
&= -4\left[-A\ln(|1-u|) + B\frac{1}{1-u} + C\ln(|1+u|) - D\frac{1}{1+u}\right] + K \\
&= -4\left[\frac{1}{4}\ln(|1-u|) + \frac{1}{4(1-u)} - \frac{1}{4}\ln(|1+u|) - \frac{1}{4(1+u)}\right] + K \\
&= -4\left[\frac{1}{4}\ln(|1-u|) + \frac{1}{4(1-u)} - \frac{1}{4}\ln(|1+u|) - \frac{1}{4(1+u)}\right] + K \\
&= -\ln(|1-u|) - \frac{1}{1-u} + \ln(|1+u|) + \frac{1}{1+u} + K \\
&= -\ln(|1-u|) + \frac{1}{u-1} + \ln(|1+u|) + \frac{1}{1+u} + K \\
&= -\ln\left(\left|1-\sqrt{\frac{1-x}{1+x}}\right|\right) + \frac{1}{\sqrt{\frac{1-x}{1+x}}-1} + \ln\left(\left|1+\sqrt{\frac{1-x}{1+x}}\right|\right) + \frac{1}{1+\sqrt{\frac{1-x}{1+x}}} + K
\end{aligned}$$

El valor de las constantes A, B, C y D se obtiene de plantear que para todo u:

$$u^2 = A(1-u)(1+u)^2 + B(1+u)^2 + C(1-u)^2(1+u) + D(1-u)^2$$

o

$$u^2 = A(1-u^2)(1+u) + B(1+u)^2 + C(1-u)(1-u^2) + D(1-u)^2$$

o

$$u^2 = A(1 + u - u^2 - u^3) + B(1 + 2u + u^2) + C(1 - u - u^2 + u^3) + D(1 - 2u + u^2)$$

es decir,

$$u^2 = (-A + C)u^3 + (-A + B - C + D)u^2 + (A + 2B - C - 2D)u + (A + B + C + D)$$

lo cual implica que:

$$\begin{aligned} -A + C &= 0 \\ -A + B - C + D &= 1 \\ A + 2B - C - 2D &= 0 \\ A + B + C + D &= 0 \end{aligned}$$

De la primera ecuación se tiene que $A = C$, sustituyendo en la tercera se obtiene que $B - D = 0$, es decir, $B = D$. Usando dichas relaciones en la cuarta ecuación se obtiene que $2A + 2B = 0$ ó $B = -A$ y por ello, $B = -C$

Ahora sustituyendo en la segunda ecuación se tiene que $-A - A - A - A = 1$, esto es, $-4A = 1$ y en consecuencia, $A = -\frac{1}{4}$, $C = -\frac{1}{4}$, $B = \frac{1}{4} = D$