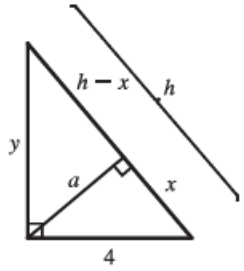
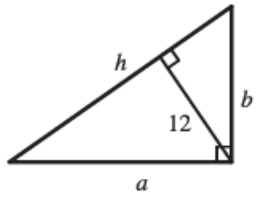
1.



By using the area formula for a triangle, $\frac{1}{2}$ (base) (height), in two ways, we see that $\frac{1}{2}(4)(y) = \frac{1}{2}(h)(a)$, so $a = \dfrac{4y}{h}$. Since $4^2 + y^2 = h^2$, $y = \sqrt{h^2 - 16}$, and $a = \dfrac{4\sqrt{h^2 - 16}}{h}$.

2.



Refer to Example 1, where we obtained $h = \dfrac{P^2 - 100}{2P}$. The 100 came from 4 times the area of the triangle. In this case, the area of the triangle is $\frac{1}{2}(h)(12) = 6h$. Thus, $h = \dfrac{P^2 - 4(6h)}{2P} \Rightarrow 2Ph = P^2 - 24h \Rightarrow 2Ph + 24h = P^2 \Rightarrow h(2P + 24) = P^2 \Rightarrow h = \dfrac{P^2}{2P + 24}$.

3. $|2x - 1| = \begin{cases} 2x - 1 & \text{if } x \geq \frac{1}{2} \\ 1 - 2x & \text{if } x < \frac{1}{2} \end{cases}$ and $|x + 5| = \begin{cases} x + 5 & \text{if } x \geq -5 \\ -x - 5 & \text{if } x < -5 \end{cases}$

Therefore, we consider the three cases $x < -5$, $-5 \leq x < \frac{1}{2}$, and $x \geq \frac{1}{2}$.

If $x < -5$, we must have $1 - 2x - (-x - 5) = 3 \Leftrightarrow x = 3$, which is false, since we are considering $x < -5$.

If $-5 \leq x < \frac{1}{2}$, we must have $1 - 2x - (x + 5) = 3 \Leftrightarrow x = -\frac{7}{3}$.

If $x \geq \frac{1}{2}$, we must have $2x - 1 - (x + 5) = 3 \Leftrightarrow x = 9$.

So the two solutions of the equation are $x = -\frac{7}{3}$ and $x = 9$.

4. $|x - 1| = \begin{cases} x - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases}$ and $|x - 3| = \begin{cases} x - 3 & \text{if } x \geq 3 \\ 3 - x & \text{if } x < 3 \end{cases}$

Therefore, we consider the three cases $x < 1$, $1 \leq x < 3$, and $x \geq 3$.

If $x < 1$, we must have $1 - x - (3 - x) \geq 5 \Leftrightarrow 0 \geq 7$, which is false.

If $1 \leq x < 3$, we must have $x - 1 - (3 - x) \geq 5 \Leftrightarrow x \geq \frac{9}{2}$, which is false because $x < 3$.

If $x \geq 3$, we must have $x - 1 - (x - 3) \geq 5 \Leftrightarrow 2 \geq 5$, which is false.

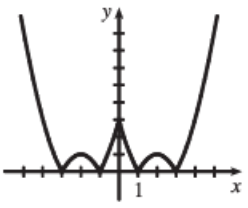
All three cases lead to falsehoods, so the inequality has no solution.

5. $f(x) = |x^2 - 4|x| + 3|$. If $x \geq 0$, then $f(x) = |x^2 - 4x + 3| = |(x - 1)(x - 3)|$.

Case (i): If $0 < x \leq 1$, then $f(x) = x^2 - 4x + 3$.

Case (ii): If $1 < x \leq 3$, then $f(x) = -(x^2 - 4x + 3) = -x^2 + 4x - 3$.

Case (iii): If $x > 3$, then $f(x) = x^2 - 4x + 3$.



This enables us to sketch the graph for $x \geq 0$. Then we use the fact that f is an even function to reflect this part of the graph about the y-axis to obtain the entire graph. Or, we could consider also the cases $x < -3$, $-3 \leq x < -1$, and $-1 \leq x < 0$.

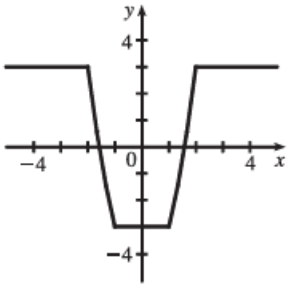
6. $g(x) = |x^2 - 1| - |x^2 - 4|$.

$$|x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } |x| \geq 1 \\ 1 - x^2 & \text{if } |x| < 1 \end{cases} \quad \text{and} \quad |x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } |x| \geq 2 \\ 4 - x^2 & \text{if } |x| < 2 \end{cases}$$

So for $0 \leq |x| < 1, g(x) = 1 - x^2 - (4 - x^2) = -3$, for

$1 \leq |x| < 2, g(x) = x^2 - 1 - (4 - x^2) = 2x^2 - 5$, and for

$|x| \geq 2, g(x) = x^2 - 1 - (x^2 - 4) = 3$.



7. Remember that $|a| = a$ if $a \geq 0$ and that $|a| = -a$ if $a < 0$. Thus,

$$x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad y + |y| = \begin{cases} 2y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

We will consider the equation $x + |x| = y + |y|$ in four cases.

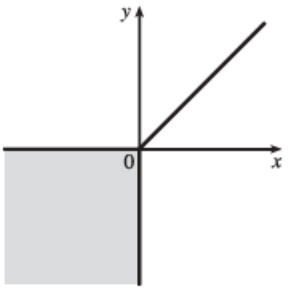
(1) $x \geq 0, y \geq 0$	(2) $x \geq 0, y < 0$	(3) $x < 0, y \geq 0$	(4) $x < 0, y < 0$
$2x = 2y$	$2x = 0$	$0 = 2y$	$0 = 0$
$x = y$	$x = 0$	$0 = y$	

Case 1 gives us the line $y = x$ with nonnegative x and y.

Case 2 gives us the portion of the y-axis with y negative.

Case 3 gives us the portion of the x-axis with x negative.

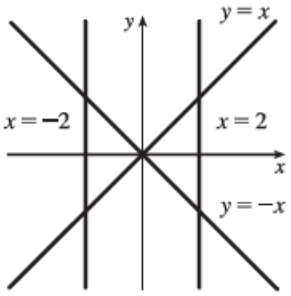
Case 4 gives us the entire third quadrant.



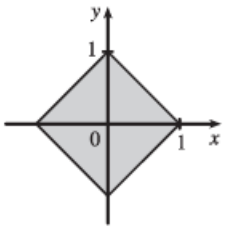
8. $x^4 - 4x^2 - x^2y^2 + 4y^2 = 0 \quad \Leftrightarrow \quad x^2(x^2 - 4) - y^2(x^2 - 4) = 0 \quad \Leftrightarrow$

$(x^2 - y^2)(x^2 - 4) = 0 \quad \Leftrightarrow \quad (x + y)(x - y)(x + 2)(x - 2) = 0$.

So the graph of the equation consists of the graphs of the four lines $y = -x$, $y = x$, $x = -2$, and $x = 2$.



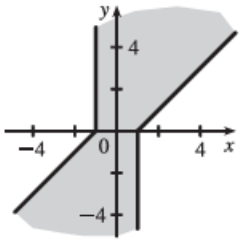
9. $|x| + |y| \leq 1$. The boundary of the region has equation $|x| + |y| = 1$. In quadrants I, II, III, and IV, this becomes the lines $x + y = 1$, $-x + y = 1$, $-x - y = 1$, and $x - y = 1$ respectively.



10. $|x-y|+|x|-|y| \le 2$

Case (i):	$x > y > 0$	$\Leftrightarrow$	$x-y+x-y \le 2$	$\Leftrightarrow$	$x-y \le 1$	$\Leftrightarrow$	$y \ge x-1$
Case (ii):	$y > x > 0$	$\Leftrightarrow$	$y-x+x-y \le 2$	$\Leftrightarrow$	$0 \le 2$ (true)		
Case (iii):	$x > 0$ and $y < 0$	$\Leftrightarrow$	$x-y+x+y \le 2$	$\Leftrightarrow$	$2x \le 2$	$\Leftrightarrow$	$x \le 1$
Case (iv):	$x < 0$ and $y > 0$	$\Leftrightarrow$	$y-x-x-y \le 2$	$\Leftrightarrow$	$-2x \le 2$	$\Leftrightarrow$	$x \ge -1$
Case (v):	$y < x < 0$	$\Leftrightarrow$	$x-y-x+y \le 2$	$\Leftrightarrow$	$0 \le 2$ (true)		
Case (vi):	$x < y < 0$	$\Leftrightarrow$	$y-x-x+y \le 2$	$\Leftrightarrow$	$y-x \le 1$	$\Leftrightarrow$	$y \le x+1$

Note: Instead of considering cases (iv), (v), and (vi), we could have noted that the region is unchanged if x and y are replaced by $-x$ and $-y$, so the region is symmetric about the origin. Therefore, we need only draw cases (i), (ii), and (iii), and rotate through $180°$ about the origin.



11. $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{31} 32) = \left(\frac{\ln 3}{\ln 2}\right)\left(\frac{\ln 4}{\ln 3}\right)\left(\frac{\ln 5}{\ln 4}\right)\cdots\left(\frac{\ln 32}{\ln 31}\right) = \frac{\ln 32}{\ln 2} = \frac{\ln 2^5}{\ln 2} = \frac{5\ln 2}{\ln 2} = 5$

12. (a) $f(-x) = \ln\left(-x+\sqrt{(-x)^2+1}\right) = \ln\left(-x+\sqrt{x^2+1}\cdot\frac{-x-\sqrt{x^2+1}}{-x-\sqrt{x^2+1}}\right)$

$= \ln\left(\frac{x^2-(x^2+1)}{-x-\sqrt{x^2+1}}\right) = \ln\left(\frac{-1}{-x-\sqrt{x^2+1}}\right) = \ln\left(\frac{1}{x+\sqrt{x^2+1}}\right)$

$= \ln 1 - \ln\left(x+\sqrt{x^2+1}\right) = -\ln\left(x+\sqrt{x^2-1}\right) = -f(x)$

(b) $y = \ln\left(x+\sqrt{x^2+1}\right)$. Interchanging x and y, we get $x = \ln\left(y+\sqrt{y^2+1}\right) \Rightarrow e^x = y+\sqrt{y^2+1} \Rightarrow$

$e^x - y = \sqrt{y^2+1} \Rightarrow e^{2x} - 2ye^x + y^2 = y^2+1 \Rightarrow e^{2x} - 1 = 2ye^x \Rightarrow y = \frac{e^{2x}-1}{2e^x} = f^{-1}(-x)$

13. $\ln\left(x^2-2x-2\right) \le 0 \Rightarrow x^2-2x-2 \le e^0 = 1 \Rightarrow x^2-2x-3 \le 0 \Rightarrow (x-3)(x+1) \le 0 \Rightarrow x \in [-1,3]$.

Since the argument must be positive, $x^2-2x-2 > 0 \Rightarrow \left[x-\left(1-\sqrt{3}\right)\right]\left[x-\left(1+\sqrt{3}\right)\right] > 0 \Rightarrow$

$x \in \left(-\infty, 1-\sqrt{3}\right) \cup \left(1+\sqrt{3}, \infty\right)$. The intersection of these intervals is $\left[-1, 1-\sqrt{3}\right) \cup \left(1+\sqrt{3}, 3\right]$.

14. Assume that $\log_2 5$ is rational. Then $\log_2 5 = m/n$ for natural numbers m and n. Changing to exponential form gives us $2^{m/n} = 5$ and then raising both sides to the nth power gives $2^m = 5^n$. But 2^m is even and 5^n is odd. We have arrived at a contradiction, so we conclude that our hypothesis, that $\log_2 5$ is rational, is false. Thus, $\log_2 5$ is irrational.

15. Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip. For the first half of the trip we have $t_1 = d/30$ and for the second half we have $t_2 = d/60$. Thus, the average speed for the entire trip is $\dfrac{\text{total distance}}{\text{total time}} = \dfrac{2d}{t_1 + t_2} = \dfrac{2d}{\dfrac{d}{30} + \dfrac{d}{60}} \cdot \dfrac{60}{60} = \dfrac{120d}{2d + d} = \dfrac{120d}{3d} = 40$. The average speed for the entire trip is 40 mi/h.

16. Let $f = \sin$, $g = x$, and $h = x$. Then the left-hand side of the equation is $f \circ (g + h) = \sin(x + x) = \sin 2x = 2 \sin x \cos x$; and the right-hand side is $f \circ g + f \circ h = \sin x + \sin x = 2 \sin x$. The two sides are not equal, so the given statement is false.

17. Let S_n be the statement that $7^n - 1$ is divisible by 6.

- S_1 is true because $7^1 - 1 = 6$ is divisible by 6.
- Assume S_k is true, that is, $7^k - 1$ is divisible by 6. In other words, $7^k - 1 = 6m$ for some positive integer m. Then $7^{k+1} - 1 = 7^k \cdot 7 - 1 = (6m + 1) \cdot 7 - 1 = 42m + 6 = 6(7m + 1)$, which is divisible by 6, so S_{k+1} is true.
- Therefore, by mathematical induction, $7^n - 1$ is divisible by 6 for every positive integer n.

18. Let S_n be the statement that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

- S_1 is true because $[2(1) - 1] = 1 = 1^2$.
- Assume S_k is true, that is, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Then
  $1 + 3 + 5 + \cdots + (2k - 1) + [(2k + 1) - 1] = 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$
  which shows that S_{k+1} is true.
- Therefore, by mathematical induction, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every positive integer n.

19. $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$.

$f_1(x) = f_0(f_0(x)) = f_0\left(x^2\right) = \left(x^2\right)^2 = x^4$, $f_2(x) = f_0(f_1(x)) = f_0(x^4) = (x^4)^2 = x^8$,

$f_3(x) = f_0(f_2(x)) = f_0(x^8) = (x^8)^2 = x^{16}, \ldots$. Thus, a general formula is $f_n(x) = x^{2^{n+1}}$.

20. (a) $f_0(x) = 1/(2-x)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$.

$$f_1(x) = f_0\left(\frac{1}{2-x}\right) = \frac{1}{2-\dfrac{1}{2-x}} = \frac{2-x}{2(2-x)-1} = \frac{2-x}{3-2x},$$

$$f_2(x) = f_0\left(\frac{2-x}{3-2x}\right) = \frac{1}{2-\dfrac{2-x}{3-2x}} = \frac{3-2x}{2(3-2x)-(2-x)} = \frac{3-2x}{4-3x},$$

$$f_3(x) = f_0\left(\frac{3-2x}{4-3x}\right) = \frac{1}{2-\dfrac{3-2x}{4-3x}} = \frac{4-3x}{2(4-3x)-(3-2x)} = \frac{4-3x}{5-4x}, \ldots$$

Thus, we conjecture that the general formula is $f_n(x) = \dfrac{n+1-nx}{n+2-(n+1)x}$.

To prove this, we use the Principle of Mathematical Induction. We have already verified that f_n is true for $n = 1$.

Assume that the formula is true for $n = k$; that is, $f_k(x) = \dfrac{k+1-kx}{k+2-(k+1)x}$. Then

$$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{k+1-kx}{k+2-(k+1)x}\right) = \frac{1}{2-\dfrac{k+1-kx}{k+2-(k+1)x}}$$

$$= \frac{k+2-(k+1)x}{2[k+2-(k+1)x]-(k+1-kx)} = \frac{k+2-(k+1)x}{k+3-(k+2)x}$$

This shows that the formula for f_n is true for $n = k+1$. Therefore, by mathematical induction, the formula is true for all positive integers n.

(b) From the graph, we can make several observations:

- The values at $x = a$ keep increasing as k increases.
- The vertical asymptote gets closer to $x = 1$ as k increases.
- The horizontal asymptote gets closer to $y = 1$ as k increases.
- The x-intercept for f_{k+1} is the value of the vertical asymptote for f_k.
- The y-intercept for f_k is the value of the horizontal asymptote for f_{k+1}.

