Interrogación 2

◁ *Viernes 14 de mayo, 2010 ⊗ 11:30-13:00* ▷

1. La posición angular θ del eje de un servomotor eléctrico satisface la ecuación diferencial:

$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + 5\theta = \frac{dr}{dt} + 3r$$

donde r es la señal de posición deseada. Si en $t = 0$ el servomotor tiene una posición angular $\theta_0 = \theta(0) = 0\,[rad]$ y una velocidad angular $\dot{\theta}_0 = \frac{d\theta}{dt}(0) = 1\,[rad/s]$:

   (a) Obtenga la representación del sistema en el espacio del estado en la forma canónica controlable.

   (b) Realice el diagrama de bloques en forma canónica observable.

   (c) Calcule la matriz de transición de estado asociada a la forma canónica controlable.

   (d) ¿Cuál será la respuesta a un impulso $r(t) = \delta(t)$?

   (e) Obtenga la posición angular $\theta(t)$ del eje si se aplica una señal $r(t) = \frac{7}{2}\left(1 - \cos\left(\frac{2\pi}{3}t\right)\right)$, para $0 \leq t < 3$, y $r(t) = 0$, para $t \geq 3$.

2. (a) Obtenga una expresión para la serie de Fourier de una onda cuadrada de periodo T para la cual el tiempo de encendido es λT, y el tiempo de apagado es $(1-\lambda)T$ con $0 \leq \lambda \leq 1$. Específicamente $x(t)$ se define como:

$$x(t) = \begin{cases} A, & kT \leq t < kT + \lambda T \\ 0, & kT + \lambda T \leq t < kT + T \end{cases}, \quad k \in \mathbb{Z}$$

   (b) Un filtro analógico descrito por la función de transferencia:

$$F(s) = \frac{s+2}{(s+10)(2s+1)}$$

   recibe como señal de entrada:

$$x(t) = 2\sin(5t) + 3\cos(7t)$$

   Obtenga el espectro bilateral de amplitud y fase de la salida $y(t)$ y calcule la energía de la salida.

3. Obtenga la transformada de Fourier de la señal $x(t) = Ap((t-d)/T)$, donde $p(t)$ representa el pulso unitario de duración 1, $A, d, T \in \mathbb{R}$ son constantes.

Tabla 1: Resumen de Trasnformadas de Laplace y Transformadas z.

$f(t)$	$F(s) = \mathcal{L}\left(f(t)\right)$	$F(z) = \mathcal{Z}\left(f[k]\right)$
$\delta(t-t_0) = \begin{cases} 1 & t = t_0 \\ 0 & t \neq t_0 \end{cases}$	$e^{-t_0 s}$	z^{-k_0}, for $t_0 = k_0 T$
$u(t-t_0) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0 \end{cases}$	$\frac{e^{-t_0 s}}{s}$	$\frac{z^{1-k_0}}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\frac{t^2}{2!}$	$\frac{1}{s^3}$	$\frac{T^2}{2}\left[\frac{z(z+1)}{(z-1)^3}\right]$
$\frac{t^3}{3!}$	$\frac{1}{s^4}$	$\frac{T^3}{6}\left[\frac{z(z^2+4z+1)}{(z-1)^4}\right]$
$\frac{t^{m-1}}{(m-1)!} = \lim_{a\to 0} \frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1} e^{-at}}{\partial a^{m-1}}$	$\frac{1}{s^m}$	$\lim_{a\to 0} \frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}\left(\frac{z}{z-e^{-aT}}\right)$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
te^{-at}	$\frac{1}{(s+a)^2}$	$Te^{-aT}z\frac{1}{(z-e^{-aT})^2}$
$\frac{t^2}{2}e^{-at}$	$\frac{1}{(s+a)^3}$	$\frac{T^2}{2}e^{-aT}z\frac{z+e^{-aT}}{(z-e^{-aT})^3}$
$\frac{t^{m-1}}{(m-1)!}e^{-at} = \frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}e^{-at}}{\partial a^{m-1}}$	$\frac{1}{(s+a)^m}$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}\left(\frac{z}{z-e^{-aT}}\right)$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{z\left(1-e^{-aT}\right)}{(z-1)(z-e^{-aT})}$
$\frac{1}{a}\left(at-1+e^{-at}\right)$	$\frac{a}{s^2(s+a)}$	$\frac{z\left[\left(aT-1+e^{-aT}\right)z+\left(1-e^{-aT}-aTe^{-aT}\right)\right]}{a(z-1)^2(z-e^{-aT})}$
$1-e^{-at}-ate^{-at}$	$\frac{a^2}{s(s+a)^2}$	$\frac{z\left[\left(1-e^{-aT}-aTe^{-aT}\right)z+\left(aT-1+e^{-aT}\right)e^{-aT}\right]}{(z-1)(z-e^{-aT})^2}$
$(1-at)e^{-at}$	$\frac{s}{(s+a)^2}$	$\frac{z\left[z-(1+aT)e^{-aT}\right]}{(z-e^{-aT})^2}$
$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$	$\frac{z\left(e^{-aT}-e^{-bT}\right)}{(z-e^{-aT})\left(z-e^{-bT}\right)}$
$be^{-bt}-ae^{-at}$	$\frac{(b-a)s}{(s+a)(s+b)}$	$\frac{z\left[(b-a)z-\left(be^{-aT}-ae^{-bT}\right)\right]}{(z-e^{-aT})\left(z-e^{-bT}\right)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\frac{z\sin(aT)}{z^2-2\cos(aT)z+1}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$\frac{z(z-\cos(aT))}{z^2-2\cos(aT)z+1}$
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$	$\frac{ze^{-aT}\sin(bT)}{z^2-2e^{-aT}\cos(bT)z+e^{-2aT}}$
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$	$\frac{z\left(z-e^{-aT}\cos(bT)\right)}{z^2-2e^{-aT}\cos(bT)z+e^{-2aT}}$
$1-e^{-at}\left(\cos(bt)+\frac{a}{b}\sin(bt)\right)$	$\frac{a^2+b^2}{s((s+a)^2+b^2)}$	$\frac{z(\alpha z+\beta)}{(z-1)(z^2-2e^{-aT}\cos(bT)z+e^{-2aT})}$
		$\alpha = 1-e^{-aT}\cos(bT)-\frac{a}{b}e^{-aT}\sin(bT)$
		$\beta = e^{-2aT}+\frac{a}{b}e^{-aT}\sin(bT)-e^{-aT}\cos(bT)$

Nota: Aquí $f[k] = f(kT)$, para $k \in \mathbb{Z}$, denota la versión discreta de f; es decir f muestreada en $t = kT$.

NOMBRE:

1. Solución Pregunta 1

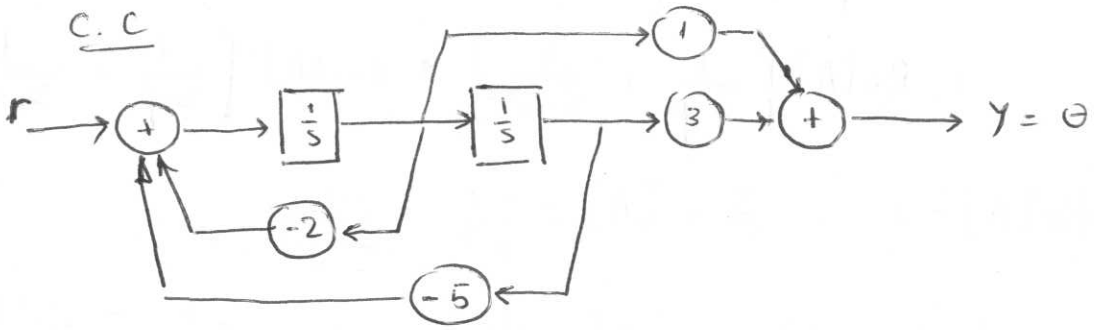
a). $(s^2 + 2s + 5)\theta = (s + 3)r$

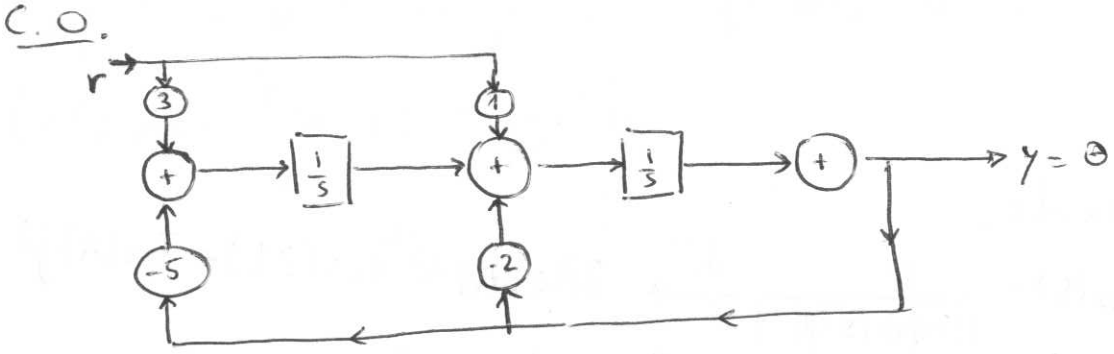
$a_1 \quad a_0 \quad b_1 \quad b_0$

$$A_c = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_c = [b_0 \quad b_1] = [3 \quad 1] \qquad D_c = [b_2] = [0]$$

b)





✓

c) $A_c = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \rightarrow \Phi(s) = (sI - A_c)^{-1} = \begin{bmatrix} s & -1 \\ 5 & s+2 \end{bmatrix}^{-1} = \dfrac{\begin{bmatrix} s+2 & 1 \\ -5 & s \end{bmatrix}}{s^2 + 2s + 5}$

$$p(s) = s^2 + 2s + 5 \rightarrow \quad s = \frac{-2 \pm \sqrt{4-20}}{2} = -\frac{2}{2} \pm j2 = -1 \pm j2$$

$$p_1 = -1 - j2$$

$$p_2 = -1 + j2$$

$$\therefore \quad m_{11}(s) = \frac{s+2}{(s-p_1)(s-p_2)} = \frac{A}{s-p_1} + \frac{B}{s-p_2}$$

$$A = m_{11}(s) \cdot (s-p_1)\Big|_{s=p_1} = \frac{p_1+2}{p_1-p_2} = \frac{1-j2}{-j4}$$

$$B = m_{11}(s)\,(s-p_2)\Big|_{s=p_2} = \frac{p_2+2}{p_2-p_1} = \frac{1+j2}{j4}$$

$$\therefore \quad m_{11}(s) = \frac{A}{s-p_1} + \frac{A^*}{s-p_1^*}$$

$$= Re[A]\left[\frac{1}{s-p_1} + \frac{1}{s-p_1^*}\right] + Im[A]\left[\frac{1}{s-p_1} - \frac{1}{s_1-p_1^*}\right]$$

$$Re[A] = \frac{1}{2}, \quad Im[A] = j\frac{1}{4} = \frac{-1}{2\cdot(j2)}$$

$$\therefore \mathcal{L}^{-1}\{m_{11}(s)\} = \frac{e^{p_1 t} + e^{p_1^* t}}{2} - \frac{1}{2}\left[\frac{e^{p_1 t} - e^{p_1^* t}}{j2}\right]$$

$$= e^{-t}\cos(2t) + \frac{e^{-t}}{2}\sin(2t)$$

Similarmente,

$$m_{12}(s) = \frac{1}{(s-p_1)(s-p_1^*)} \xrightarrow{\mathcal{L}^{-1}} 2Re[A]\cdot e^{-t}\cos(2t) + Im[A]\cdot j2\, e^{-t}\sin(2t)$$

$$A = \frac{1}{p_1 - p_2} = \frac{1}{-j4}$$

$$\therefore \quad m_{12}(t) = +\frac{1}{2}e^{-t}\sin(2t)$$

$$m_{21}(s) = \frac{-5}{(s-p_1)(s-p_1^*)} \xrightarrow{\mathcal{L}^{-1}} m_{21}(t) = \frac{5}{2}e^{-t}\sin(2t)$$

$$M_{22}(s) = s \cdot \underbrace{M_{12}(s)}_{\mathcal{L}^{-1}} \xrightarrow{\mathcal{L}^{-1}} m_{22}(t) = \frac{d}{dt} m_{12}(t)$$

$$= s\left[+\frac{1}{2} e^{-t}\sin(2t)\right] = -\frac{1}{2} e^{-t}\sin(2t) + e^{-t}\cos(2t)$$

o bien

$$M_{22}(s) = \frac{s}{(s-p_1)(s-p_1^*)} \xrightarrow{\mathcal{L}^{-1}} 2\,\mathrm{Re}[A]\, e^{-t}\cos(2t) - \mathrm{Im}[A]\, j2\, e^{-t}\sin(2t)$$

$$A = \frac{p_1}{p_1 - p_1^*} = \frac{-1-j2}{-j4} = \frac{2-j}{4} = \frac{1}{2} - \frac{j}{4}$$

$$\therefore \quad m_{22}(t) = e^{-t}\cos(2t) + \frac{1}{2} e^{-t}\sin(2t)$$

d) $$y(t) = C e^{At} x(0) + C \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$u(t) = \delta(t)$

$x(0) = 0 \longrightarrow y(t) = h(t) = C e^{At} B$

$$= [3 \;\; 1]\, e^{A_c t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{3}{2} e^{-t}\sin(2t) + e^{-t}\cos(2t) - \frac{1}{2} e^{-t}\sin(2t)$$

$$= e^{-t}\left[\sin(2t) + \cos(2t)\right]$$

e) $y(t) = C e^{At} x(0) + C \int_0^t e^{A(t-\tau)} B r(\tau)\, d\tau$

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} x(0)$$

$$\longrightarrow \quad x(0) = \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} \theta(0) \\ \dot{\theta}(0) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ -5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix}}{8} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/8 \\ 3/8 \end{bmatrix}$$

$$y_p(t) = h(t) * r(t)$$

$$= \int_0^t \left[e^{-(t-\tau)} \sin(2(t-\tau)) + e^{-(t-\tau)} \cos(2(t-\tau)) \right] \frac{7}{2} \left(1 - \cos\left(\frac{2\pi}{3}\tau\right)\right) d\tau$$

$$= +\frac{7}{2} \left(18\pi^2 e^{-t} \sin(t) - 4\pi^4 e^{-t} \cos(t) - 27\pi \sin\left(\frac{2}{3}\pi t\right) \right.$$

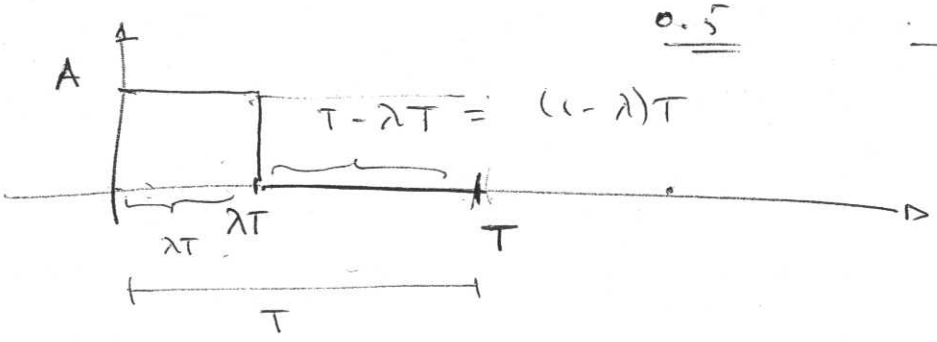
$$\left. -6\pi^3 \sin\left(\frac{2\pi}{3}t\right) - 81 \cos\left(\frac{2\pi}{3}t\right) + 81 + 4\pi^4 \right) \frac{1}{(9 + 6\pi + 2\pi^2)} \cdot$$

$$\cdot \frac{1}{(9 - 6\pi + 2\pi^2)}, \quad 0 \le t \le 3$$

$$y_h(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} e^{At} \begin{bmatrix} -1/8 \\ 3/8 \end{bmatrix} \quad t \ge 0$$

NOMBRE:

2. Solución Pregunta 2



0.5

k		f
0	0, λT	A
	λT, T	0
1	T, λT+T	A
	λT+T, 2T	0
2		

$$c_n = \frac{1}{T}\int_{<T>} x(t)\cdot e^{-jn\omega_0 t}\,dt$$

0.5

$\omega_0 = \frac{2\pi}{T}$

$$= \frac{1}{T}\int_0^T x(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^{\lambda T} A\cdot e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\,\frac{1}{-jn\omega_0}\cdot e^{-jn\omega_0 t}\Big|_{t=0}^{\lambda T}$$

$$= \frac{A}{T}\cdot\frac{1}{-jn\omega_0}\cdot\left[e^{-j\omega_0 n\lambda T} - 1\right]$$

$\omega_0 = \frac{2\pi}{T}$ $\qquad$ $\omega_0 = \frac{2\pi}{T}$

1.5

$$= \frac{jA}{2\pi n}\left[\underbrace{e^{-j2\pi n\lambda}}_{\cos(2\pi n\lambda) + j\sin(2\pi n\lambda)} - 1\right]$$

$\lim_{n\to\infty} A\,2\pi\lambda\,\frac{\cos(2\pi n\lambda)}{2\pi}$

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T}\cdot A\cdot\lambda T = \lambda A$$

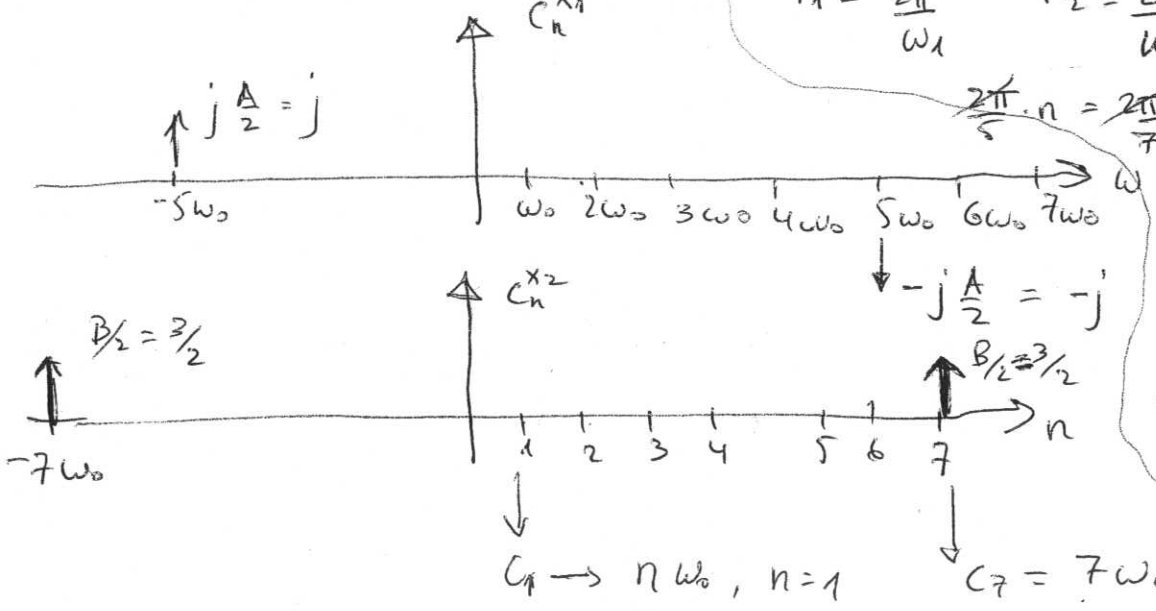
0.5

x_1 x_2

b). $x(t) = \underset{A}{2}\sin(5t) + \underset{B}{3}\cos(7t)$

0.3

$\omega_0 = 1$ rad/s



$T_1 \cdot n = T_2 \cdot m$

$T_1 = \frac{2\pi}{\omega_1}$ $T_2 = \frac{2\pi}{\omega_2}$

$\frac{2\pi}{5} \cdot n = \frac{2\pi}{7} \cdot m$

$7n = 5m$

$m = 7$

$n = 5$

$T_f = nT_1 = mT_2$

$= 5 \cdot \frac{2\pi}{5} = 2\pi$

$\omega_f = \frac{2\pi}{T_f} = 1$

$$c_n^y = \sum_{n=-\infty}^{\infty} H(n\omega_0) \cdot c_n^x$$

0.2

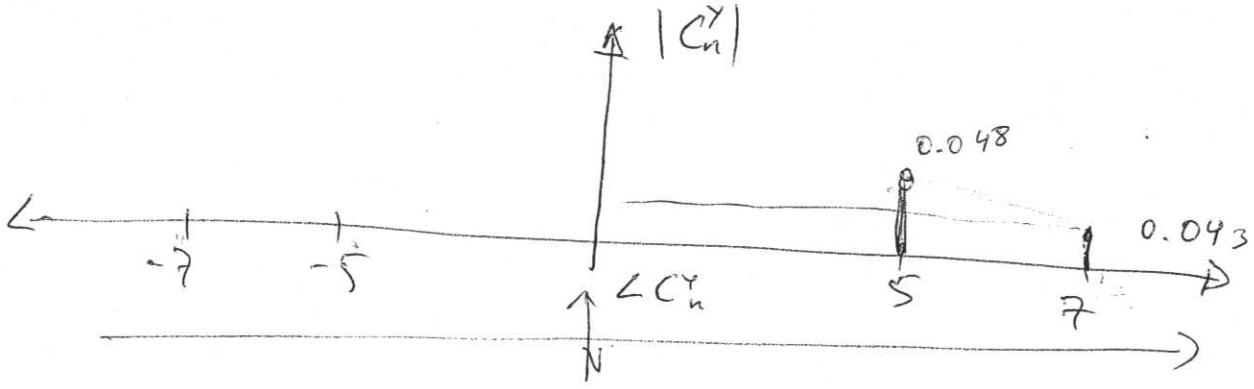
$$H(n\omega_0) = \frac{jn\omega_0 + 2}{(jn\omega_0 + 10)(2jn\omega_0 + 1)}$$

0.5

$\omega_0 = 1$ rad/s

$$H(n\omega_0) \equiv H(n) = \frac{jn+2}{(jn+10)(j2n+1)}$$

$$= \frac{j(-7)+2}{(-j\cdot 7+10)(-j14+1)} \cdot \frac{3}{2} + \frac{-j5+2}{(-j5+10)(-j10+1)} \cdot j$$



0.5 dibujo.

$$C_5^Y = H(5)\cdot C_5^X = \frac{j5+2}{(j5+10)(j10+1)}\cdot(-j)$$

$$C_7^Y = H(7)\cdot C_7^X = \frac{j7+2}{(j7+10)\cdot(j14+1)}\cdot\frac{3}{2}$$

$$|C_5^Y| = \frac{\sqrt{25+4}}{\sqrt{25+100}\cdot\sqrt{100+1}} = \frac{\sqrt{29}}{\sqrt{101\cdot125}} = \sqrt{\frac{29}{101\cdot125}}$$

$$\simeq \frac{\cancel{30}\ \cancel{6}\ 3}{\cancel{20}\ \cancel{100}\cdot125} = \frac{1}{16}$$
$$10\quad 5$$

$$|C_7^Y| = \frac{\sqrt{49+4}}{\sqrt{49+100}\cdot\sqrt{196+1}} = \sqrt{\frac{53}{149\cdot187}} \simeq \frac{7\ldots}{150}$$

$$\frac{1.7}{3.3\cdot11}\quad\frac{2}{3\cdot11}\quad\frac{2}{33}\quad\frac{1}{16}$$

0.5
$$\begin{cases}\angle C_5^Y = \operatorname{atan}(5/2) - \operatorname{atan}(5/10) - \operatorname{atan}(10) = \ldots \\ \angle C_7^Y = \operatorname{atan}(7/2) - \operatorname{atan}(7/10) - \operatorname{atan}(14) = \ldots\end{cases}$$
$$14\quad 14\quad 196$$

$$C_{-5}^Y = -C_5^Y \quad , \quad C_{-7}^Y = -C_7^Y$$

0.5 $E_y = \sum_{n=-\infty}^{\infty} C_n^{y\,2} = \frac{2.29}{101.125} + \frac{2.53}{149.187}$

NOMBRE:

3. Solución Pregunta 3

$$x(t) = A\, p\left(\frac{(t-d)}{T}\right) = A\, p\left(\frac{t}{T} - \frac{d}{T}\right)$$

i) Usando las propiedades

$$\mathcal{F}\{f(t-t_0)\} = e^{-j\omega t_0} F(\omega)$$

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F(a\omega)$$

$$\mathcal{F}\left\{p\left(\frac{t}{\tau}\right)\right\} = \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right) = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$

con $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$

$a\omega$ $\quad -t_0$

$$\mathcal{F}\{x(t)\} = X(\omega) = A e^{j(T\omega)\cdot\left(\frac{-d}{T}\right)} \cdot \frac{2}{\omega}\cdot\sin\left(\frac{\omega T}{2}\right)$$

$$= A e^{-j\omega d}\, T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right) \quad \square$$

ii) Resolviendo según definición (Esto era lo que había que hacer).

$$X(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} A\, p\left(\frac{(t-d)}{T}\right) \cdot e^{-j\omega t}\, dt$$

$$p\left(\frac{t-d}{T}\right) = \begin{cases} 1 & , \left|\frac{t-d}{T}\right| \leq \frac{1}{2} \rightarrow t \in \left[d - \frac{T}{2},\ d + \frac{T}{2}\right] \\ 0 & , \sim \end{cases}$$

$$\therefore X(\omega) = A \int_{d-T/2}^{d+T/2} e^{-j\omega t}\, dt$$

$$= A \cdot \frac{-1}{j\omega} \cdot e^{-j\omega t} \Big|_{t=d-T/2}^{d+T/2}$$

$$= A\, \frac{-1}{j\omega} \left[e^{-j\omega\left(d+\frac{T}{2}\right)} - e^{-j\omega\left(d-\frac{T}{2}\right)} \right]$$

$$= e^{-j\omega d}\, \frac{2A}{\omega} \cdot \left[\frac{e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}{j2} \right]$$

$$= e^{-j\omega d}\, \frac{2A}{\omega} \sin\left(\frac{\omega T}{2}\right) = e^{-j\omega d}\, AT \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

□