IEE 2103 Señales y Sistemas

Clase 6G: Transformada de Fourier Discreta

FFT

6.8 Transformada Rápida de Fourier

FFT: Fast Fourier Transform

La idea es subdividir el problema en dos y tomarle la DFT a cada una de las partes para luego combinarlas.

Se reduce el tiempo de computación de $O(N^2)$ a $O(N \log N)$

Paper original sobre FFT

An Algorithm for the Machine Calculation of Complex Fourier Series
Author(s): James W. Cooley and John W. Tukey
Source: *Mathematics of Computation*, Vol. 19, No. 90 (Apr., 1965), pp. 297-301
Published by: American Mathematical Society



Paper original sobre FFT

An Algorithm for the Machine Calculation of Complex Fourier Series

By James W. Cooley and John W. Tukey

An efficient method for the calculation of the interactions of a 2^m factorial experiment was introduced by Yates and is widely known by his name. The generalization to 3^m was given by Box et al. [1]. Good [2] generalized these methods and gave elegant algorithms for which one class of applications is the calculation of Fourier series. In their full generality, Good's methods are applicable to certain problems in which one must multiply an N-vector by an $N \times N$ matrix which can be factored into m sparse matrices, where m is proportional to $\log N$. This results in a procedure requiring a number of operations proportional to $N \log N$ rather than N^2. These methods are applied here to the calculation of complex Fourier series. They are useful in situations where the number of data points is, or can be chosen to be, a highly composite number. The algorithm is here derived and presented in a rather different form. Attention is given to the choice of N. It is also shown how special advantage can be obtained in the use of a binary computer with $N = 2^m$ and how the entire calculation can be performed within the array of N data storage locations used for the given Fourier coefficients.

Consider the problem of calculating the complex Fourier series

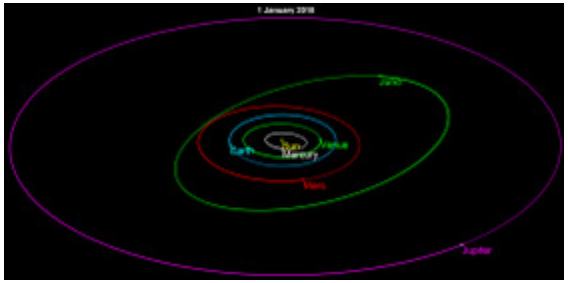
$$X(j) = \sum_{k=0}^{N-1} A(k) \cdot W^{jk}, \qquad j = 0, 1, \cdots, N-1, \tag{1}$$

where the given Fourier coefficients $A(k)$ are complex and W is the principal Nth root of unity,

$$W = e^{2\pi i/N} \tag{2}$$

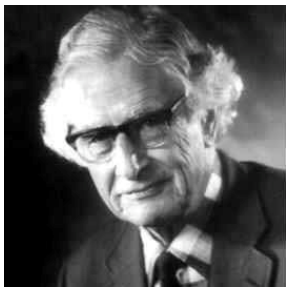
Historia de la FFT

This algorithm, including its recursive application, was invented around 1805 by Carl Friedrich Gauss, who used it to interpolate the trajectories of the asteroids Pallas and Juno, but his work was not widely recognized (being published only posthumously and in neo-Latin). Gauss did not analyze the asymptotic computational time, however.





Various limited forms were also rediscovered several times throughout the 19th and early 20th centuries. FFTs became popular after James Cooley of IBM and John Tukey of Princeton published a paper in 1965 reinventing the algorithm and describing how to perform it conveniently on a computer.



Frank Yates (1932)

Directo de Wikipedia

Historia de la FFT

Tukey reportedly came up with the idea during a meeting of President Kennedy's Science Advisory Committee discussing ways to detect nuclear-weapon tests in the Soviet Union by employing seismometers located outside the country. These sensors would generate seismological time series. However, analysis of this data would require fast algorithms for computing DFT due to number of sensors and length of time. This task was critical for the ratification of the proposed nuclear test ban so that any violations could be detected without need to visit Soviet facilities.



Another participant at that meeting, Richard Garwin of IBM, recognized the potential of the method and put Tukey in touch with Cooley however making sure that **Cooley did not know the original purpose**. Instead Cooley was told that this was needed to determine periodicities of the spin orientations in a 3-D crystal of Helium-3. Cooley and Tukey subsequently published their joint paper, and wide adoption quickly followed due to the simultaneous development of Analog-to-digital converters capable of sampling at rates up to 300 kHz.

Directo de Wikipedia

Paper original sobre FFT

Tukey Neologisms

John Tukey was famous for creating new words and new uses for old ones. He said that he did this to avoid confusions.

He is said to have introduced the terms: "bit", "linear programming", "ANOVA", "Colonel Blotto", and was first into print with "software". Of these efforts L. Hogben and M. Cartwright wrote, "The introduction by Tukey of bits for binary digits has nothing but irresponsible vulgarity to commend it." Tukey's word "polykay" was described as "linguistic miscegenation" by Kendall and Stuart because of its combining a Greek prefix with a Latin suffix. JWT did it again later with "polyspectrum".

6.8 Transformada Rápida de Fourier

Subdividiendo el vector de entrada en elementos pares e impares se tiene

$$\tilde{X}[k] = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-i2\pi kn/N}$$

$$\tilde{X}[k] = \frac{1}{N} \left(\sum_{n \text{ par}} \tilde{x}[n] e^{-i2\pi kn/N} + \sum_{n \text{ impar}} \tilde{x}[n] e^{-i2\pi kn/N} \right)$$

6.8 Transformada Rápida de Fourier

$$\begin{aligned}\tilde{X}[k] &= \frac{1}{N}\left(\sum_{r=0}^{\frac{N}{2}-1}\tilde{x}[2r]e^{-i4\pi kr/N} + \sum_{r=0}^{\frac{N}{2}-1}\tilde{x}[2r+1]e^{-i2\pi k(2r+1)/N}\right) \\ &= \frac{1}{N}\left(\sum_{r=0}^{\frac{N}{2}-1}\tilde{x}[2r]e^{-i4\pi kr/N} + e^{-i2\pi k/N}\sum_{r=0}^{\frac{N}{2}-1}\tilde{x}[2r+1]e^{-i4\pi kr/N}\right)\end{aligned}$$

Si llamamos

$$e^{-i2\pi/N} = W_N$$

6.8 Transformada Rápida de Fourier

$$\tilde{X}[k] = \frac{1}{N}\Big(\underbrace{\sum_{r=0}^{\frac{N}{2}-1} \tilde{x}[2r] W_{N/2}^{kr}}_{\frac{N}{2}\tilde{G}[k]} + W_N^k \underbrace{\sum_{r=0}^{\frac{N}{2}-1} \tilde{x}[2r+1] W_{N/2}^{kr}}_{\frac{N}{2}\tilde{H}[k]}\Big)$$

DFT de elementos pares

DFT de elementos impares

$$2\tilde{X}[k] = \tilde{G}[k] + W_N^k \tilde{H}[k].$$

uf!

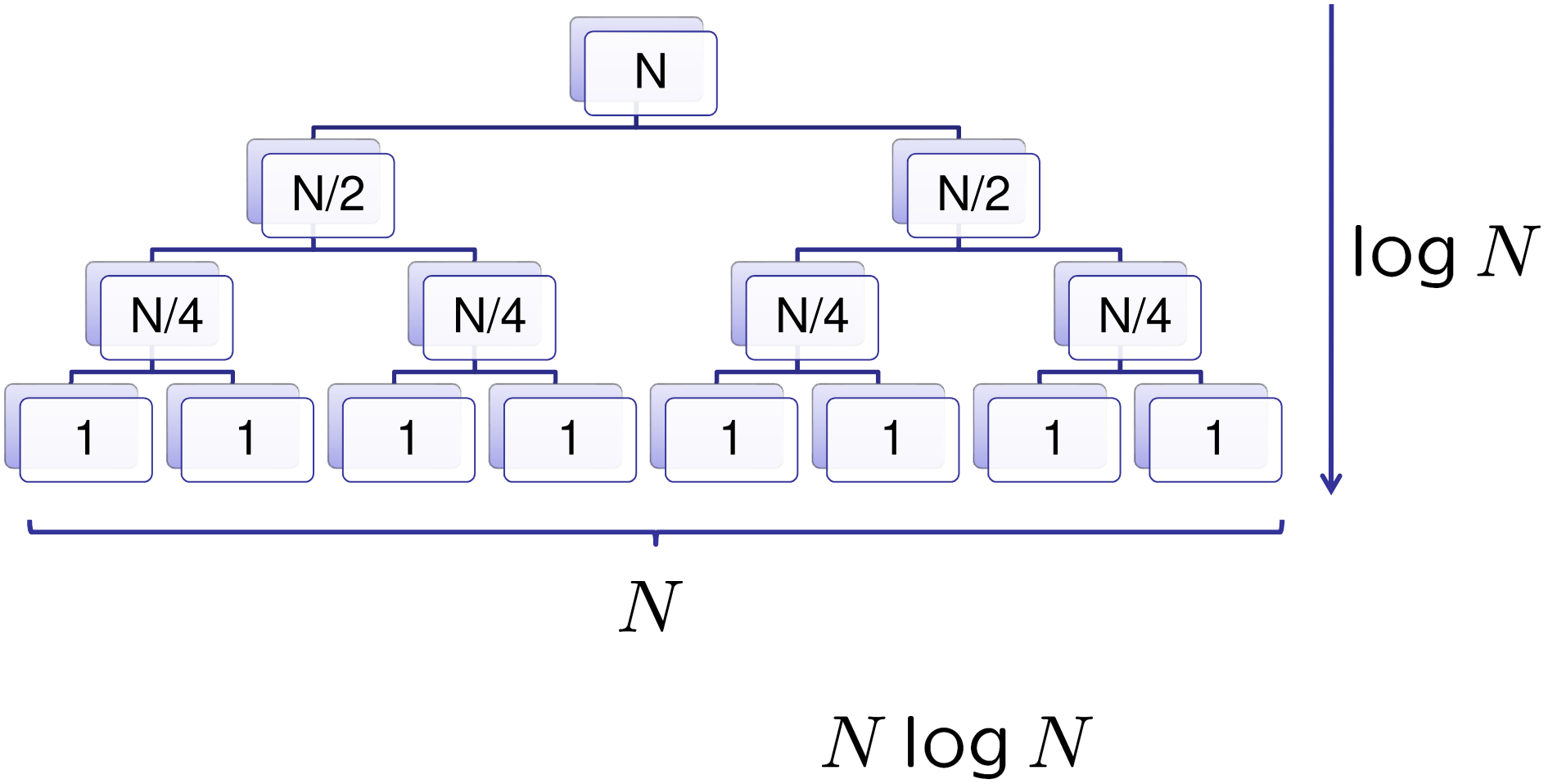
6.8 Transformada Rápida de Fourier

$$2\tilde{X}[k] = \tilde{G}[k] + W_N^k \tilde{H}[k].$$

2 DFTs de N/2 puntos

DFT de N puntos

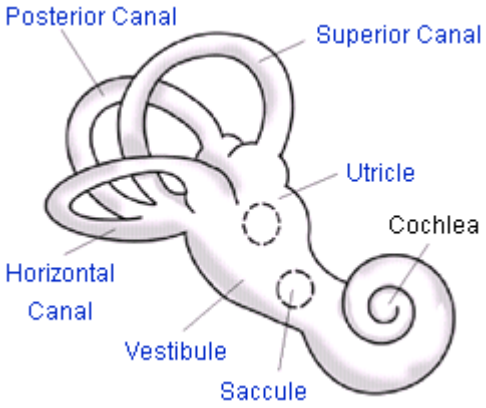
6.8 Transformada Rápida de Fourier

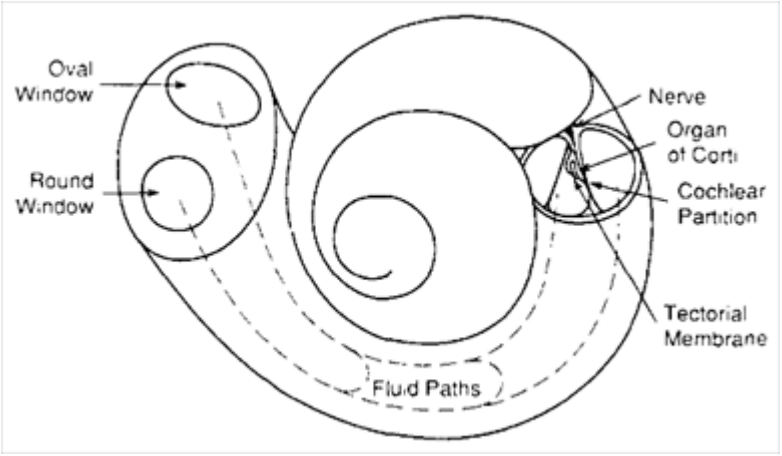


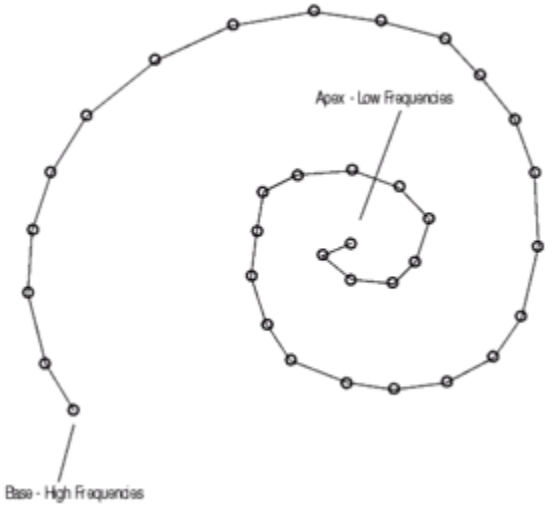
Ejemplo de aplicación

Implante coclear

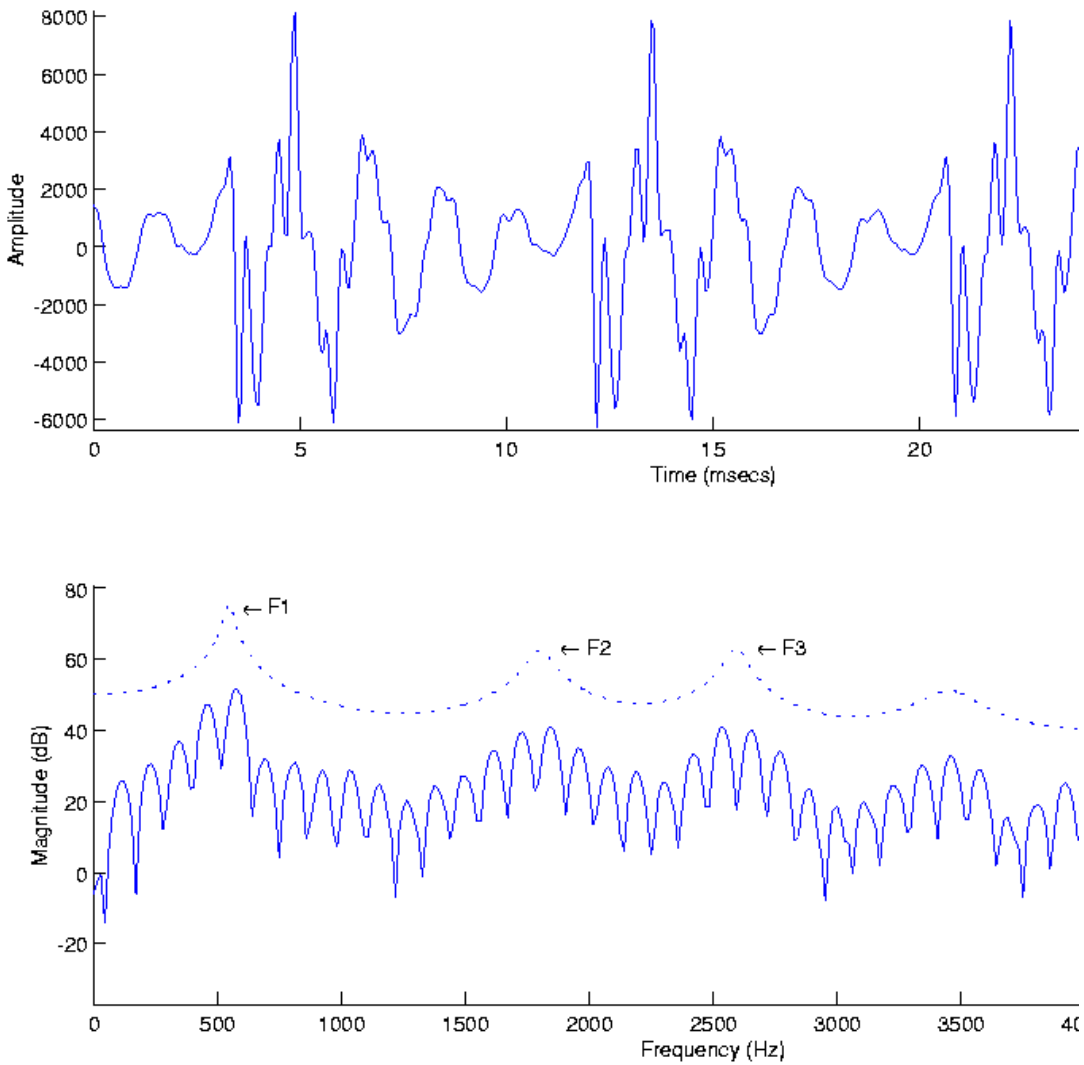
Coclea


Posterior Canal
Superior Canal
Utricle
Cochlea
Horizontal
Canal
Vestibule
Saccule

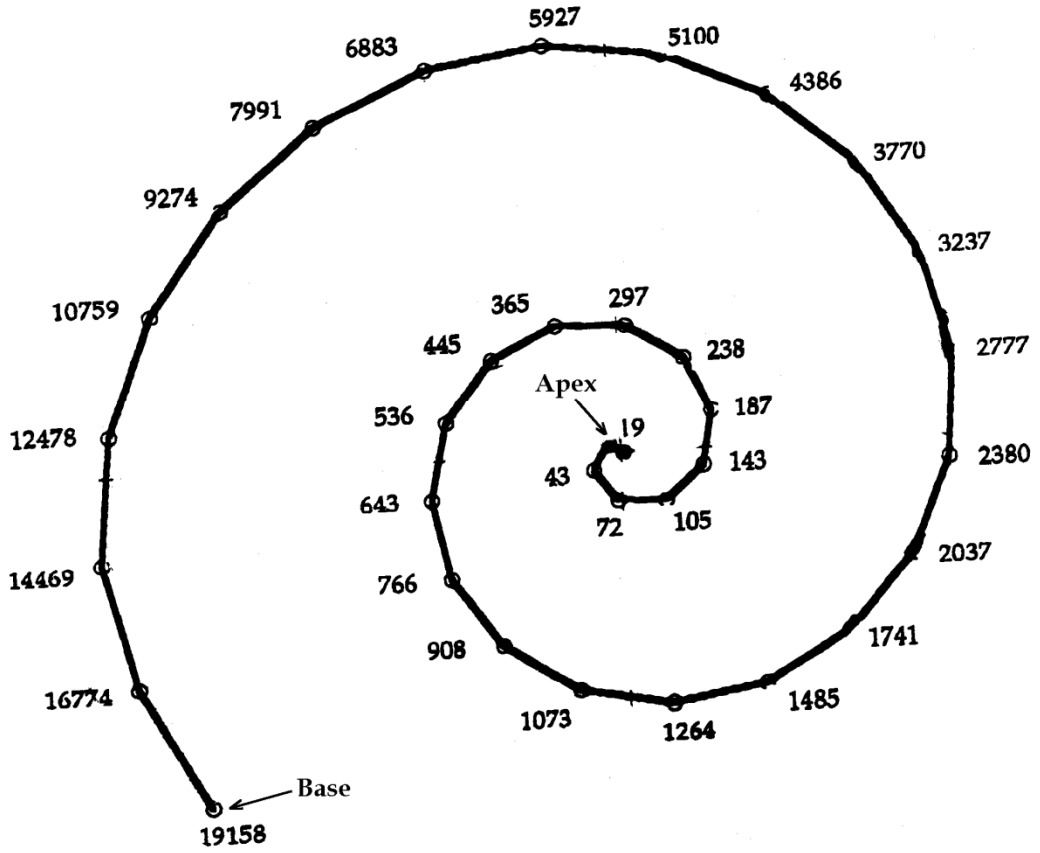

Oval
Window
Round
Window
Nerve
Organ
of Corti
Cochlear
Partition
Tectorial
Membrane
Fluid Paths


Apex - Low Frequencies
Base - High Frequencies

Sensibilidad Coclea

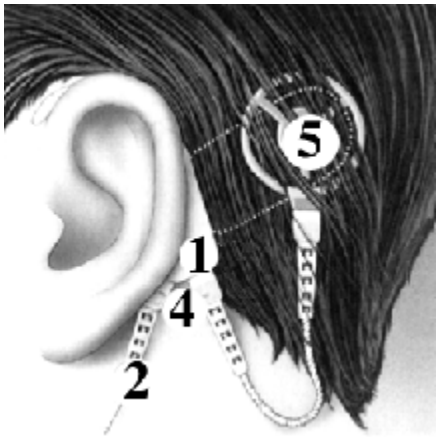


Sensibilidad Coclea

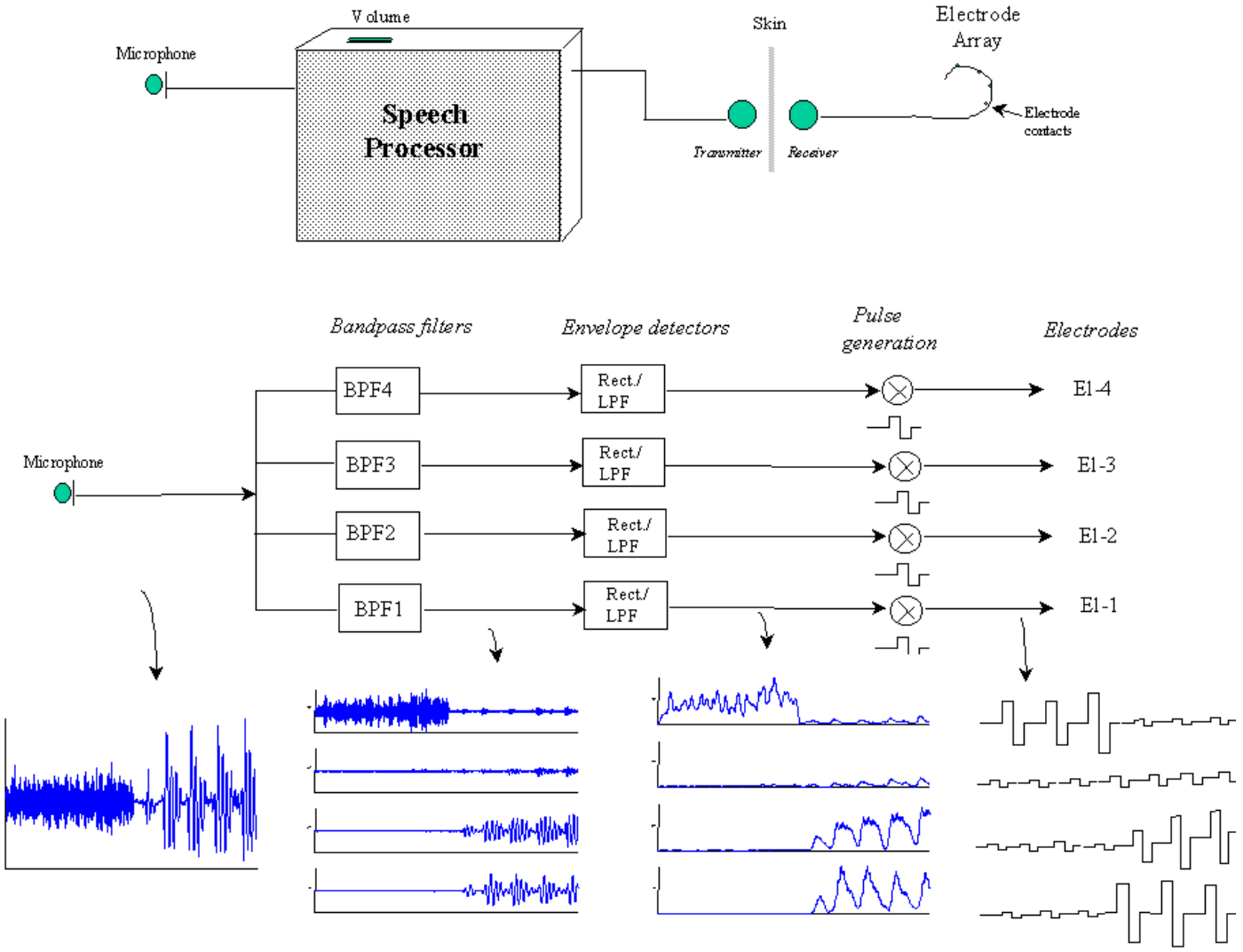


Implante Coclear

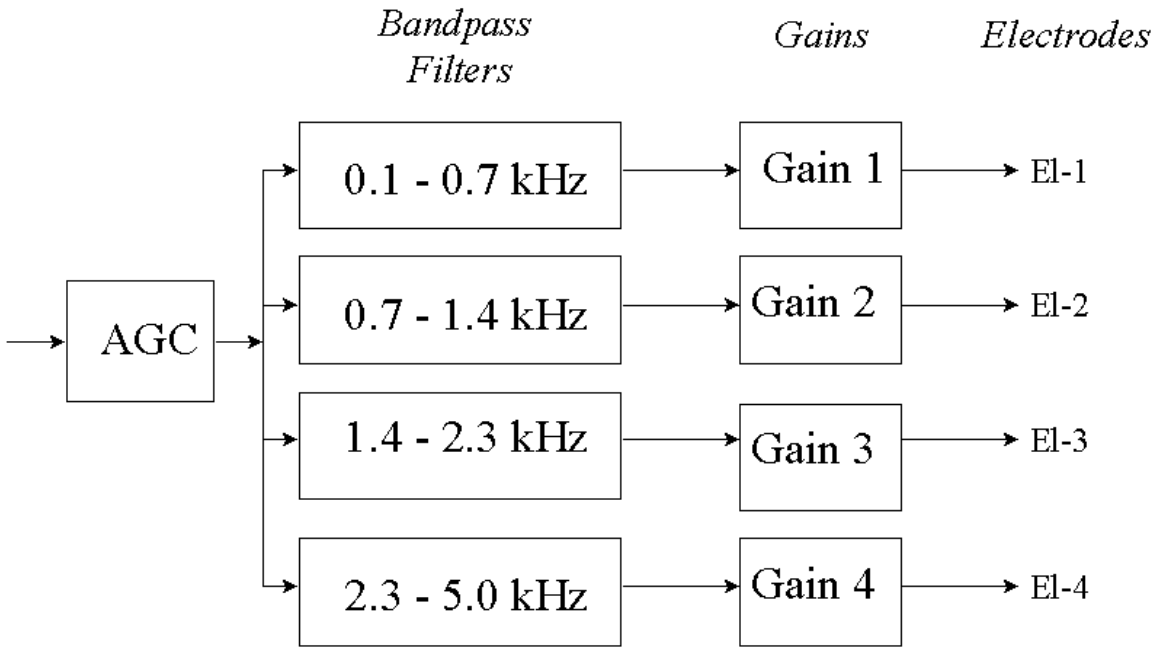




Implante Coclear



Implante Coclear



Implante Coclear

